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AUTHOR(S):

Kanoko, Tomoaki

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# Transcendence of certain reciprocal sums of linear recurrences

慶応大・理工学研究科

鹿子 智朗 (Tomoaki Kanoko)

Science and Technology,  
Keio Univ.

## 1 Introduction

Let  $C$  be a field of characteristic 0 and  $d$  an integer greater than 1. We consider the function  $f(z)$  defined by

$$f(z) = \sum_{k \geq 0} \frac{a^k z^{d^k}}{H(z^{d^k})}, \quad (1)$$

where  $H(z) \in C[z]$  with  $H(0) = 1$  and  $\deg H(z) \geq 1$ , and  $a \in C$  with  $a \neq 0$ . Then the function  $f(z)$  satisfies the functional equation

$$af(z^d) = f(z) - \frac{z}{H(z)}. \quad (2)$$

It is known that  $f(z)$  represents a rational function in the following four cases:

(i) If  $d = 2, a = 4$ , and  $H(z) = (1 + z)^2$ , then

$$f(z) = \sum_{k \geq 0} \frac{4^k z^{2^k}}{(1 + z^{2^k})^2} = \frac{z}{(1 - z)^2}.$$

(ii) If  $d = 2, a = -2$ , and  $H(z) = 1 - z + z^2$ , then

$$f(z) = \sum_{k \geq 0} \frac{(-2)^k z^{2^k}}{1 - z^{2^k} + z^{2^{k+1}}} = \frac{z}{1 + z + z^2}.$$

(iii) If  $d = 2, a = 2$ , and  $H(z) = 1 + z$ , then

$$f(z) = \sum_{k \geq 0} \frac{2^k z^{2^k}}{1 + z^{2^k}} = \frac{z}{1 - z}.$$

(iv) If  $d = 2$ ,  $a = 1$ , and  $H(z) = 1 - z^2$ , then

$$f(z) = \sum_{k \geq 0} \frac{z^{2^k}}{1 - z^{2^{k+1}}} = \frac{z}{1 - z}.$$

It is natural to ask whether there exist rational functions of the form (1) other than these four cases. The purpose of this paper is to answer this question.

**Theorem 1.1.** *Let  $f(z)$  be the function defined by (1). Suppose that  $\deg H \leq 3$ . Then  $f(z)$  is a transcendental function over  $C(z)$  except in the four cases stated above.*

In the case of  $a \neq 1$ , we can dispense with the assumption  $\deg H \leq 3$ .

**Theorem 1.2.** *Let  $f(z)$  be the function defined by (1). Suppose that  $a \neq 1$ . Then  $f(z)$  is a transcendental function over  $C(z)$  except in the three cases stated above.*

We shall apply Theorem 1.1 to establish the transcendence of new type of reciprocal sums of binary linear recurrences.

Let  $\{F_n\}_{n \geq 0}$  be the sequence of the Fibonacci numbers defined by

$$F_0 = 0, F_1 = 1, F_{n+2} = F_{n+1} + F_n \quad (n \geq 0),$$

and  $\{L_n\}_{n \geq 0}$  be the sequence of the Lucas numbers defined by

$$L_0 = 2, L_1 = 1, L_{n+2} = L_{n+1} + L_n \quad (n \geq 0).$$

Lucas [6] proved that

$$\theta_1 = \sum_{k \geq 0} \frac{1}{F_{2^k}} = \frac{7 - \sqrt{5}}{2}.$$

Erdős and Graham [5] asked for arithmetic character of the related sums

$$\theta_2 = \sum_{k \geq 0} \frac{1}{L_{2^k}}, \quad \theta_3 = \sum_{k \geq 0} \frac{1}{F_{2^k+1}}.$$

Transcendence of  $\theta_2$  and that of  $\theta_3$  were proved by Bundschuh and Pethö [2] and by Becker and Töpfer [1], respectively.

Let  $\{R_n\}_{n \geq 0}$  be a sequence of integers satisfying the binary linear recurrence relation

$$R_{n+2} = A_1 R_{n+1} + A_2 R_n \quad (n \geq 0), \tag{3}$$

where  $A_1 \neq 0$ ,  $A_2$  are integers,  $\Delta = A_1^2 + 4A_2 > 0$  is not a perfect square, and  $R_0, R_1$  are integers not both zero. We can express  $\{R_n\}_{n \geq 0}$  as follows:

$$R_n = g_1 \alpha^n + g_2 \beta^n \quad (n \geq 0),$$

where  $g_1 = (R_1 - \beta R_0)/(\alpha - \beta)$ ,  $g_2 = (\alpha R_0 - R_1)/(\alpha - \beta)$ , and  $\alpha, \beta$  are the roots of

$$X^2 - A_1 X - A_2 = 0.$$

Then we define  $R_l$  for any  $l \in \mathbb{Z}$  by  $R_l = g_1 \alpha^l + g_2 \beta^l$ .

Becker and Töpfer [1] proved a more general theorem.

**Theorem A (Becker and Töpfer [1]).** *Let  $\{R_n\}_{n \geq 0}$  be a sequence of integers satisfying (3),  $\{a_k\}_{k \geq 0}$  be a periodic sequence of algebraic numbers which is not identically zero, and  $d, c$ , and  $l$  be integers with  $d \geq 2$  and  $c \geq 1$ . Then the number*

$$\theta = \sum'_{k \geq 0} \frac{a_k}{R_{cd^k+l}},$$

where the sum  $\sum'_{k \geq 0}$  is taken over those  $k$  with  $cd^k + l \geq 0$  and  $R_{cd^k+l} + b \neq 0$ , is algebraic if and only if  $\{a_k\}_{k \geq 0}$  is a constant sequence,  $d = 2$ ,  $|A_2| = 1$ , and  $R_l = 0$ .

Their result was much more improved by Nishioka, Tanaka, and Toshimitu [10]. Indeed they established the algebraic independence of the numbers

$$\sum'_{k \geq 0} \frac{a_k}{(R_{cd^k+l})^m} \quad (d \geq 2, m \geq 1, l \in \mathbb{Z})$$

even under a weaker condition on  $\{R_n\}_{n \geq 0}$ .

Duverney [3] showed that

$$\sum_{k \geq 1} \frac{4^k}{L_{2^k} + 2} = 4, \quad \sum_{k \geq 1} \frac{(-2)^k}{L_{2^k} - 1} = -\frac{1}{2}. \quad (4)$$

These numbers are special cases of the following reciprocal sums

$$\phi = \sum'_{k \geq 0} \frac{a_k}{R_{cd^k+l} + b}, \quad (5)$$

where the sum  $\sum'_{k \geq 0}$  is taken over those  $k$  with  $cd^k + l \geq 0$  and  $R_{cd^k+l} + b \neq 0$ ,  $\{a_k\}_{k \geq 0}$  is a linear recurrence of algebraic numbers which is not identically zero, and  $b, c, d$ , and  $l$  are integers with  $c \geq 1$  and  $d \geq 2$ . Using Theorem 1.1 and applying a method developed in [9], we can show that these numbers are transcendental except some few cases including the numbers given by (4).

**Theorem 1.3.** *Let  $\{R_n\}_{n \geq 0}$  be a sequence of integers satisfying (3). Then the number  $\phi$  defined by (5) is transcendental except in the following three cases:*

- (i)  $|A_2| = 1, d = 2, b = 0, R_l = 0$ , and  $\{a_k\}_{k \geq 0}$  is a constant sequence.
- (ii)  $|A_2| = 1, d = 2, A_1 R_l = 2R_{l+1}, R_l = b$ , and  $a_k = c4^k$  ( $k \geq 0$ ) for some nonzero  $c \in \overline{\mathbb{Q}}$ .
- (iii)  $|A_2| = 1, d = 2, A_1 R_l = 2R_{l+1}, R_l = -2b$ , and  $a_k = c(-2)^k$  ( $k \geq 0$ ) for some nonzero  $c \in \overline{\mathbb{Q}}$ .

**Remark 1.1.** Becker and Töpfer's result stated above can be deduced from Theorem 1.3.

## 2 Proof of Theorems

### 2.1 Proof of Theorem 1.1

The function  $f(z)$  is transcendental over  $C(z)$  if  $f(z) \notin C(z)$  (cf. [7]). Suppose on the contrary that  $f(z) = P(z)/Q(z)$  with  $P(z), Q(z) \in C[z]$  prime to each other. As  $f(0) = 0$ , we have  $P(0) = 0$  and  $Q(0) \neq 0$ , so that we may assume  $Q(0) = 1$ . By (2) we have

$$a \frac{P(z^d)}{Q(z^d)} = \frac{P(z)}{Q(z)} - \frac{z}{H(z)},$$

and so

$$aP(z^d)Q(z)H(z) = P(z)Q(z^d)H(z) - zQ(z)Q(z^d). \quad (6)$$

As  $P(z^d)/Q(z^d)$  is irreducible,  $Q(z^d)$  divides  $Q(z)H(z)$ . Therefore there exist  $A(z) \in C[z]$  such that

$$A(z)Q(z^d) = Q(z)H(z). \quad (7)$$

As  $P(0) = 0$ , we put  $P(z) = zR(z)$ . We have from (6)

$$Q(z)^2 = A(z)\{R(z)Q(z^d) - az^{d-1}R(z^d)Q(z)\}. \quad (8)$$

In what follows, let  $h, p, q$ , and  $r$  be the degrees of  $H, P, Q$ , and  $R$ , respectively. Then we have by (7) and (8)

$$\deg A = h - (d-1)q \leq 2q. \quad (9)$$

We shall prove

$$1 \leq 1 + r = p \leq q. \quad (10)$$

As  $P(0) = 0$ , we have  $p \geq 1$ . If  $p > q$ , we get  $\deg PH > \deg zQ$ , since  $1 \leq h (\leq 3)$  by (2) and (7). Then (6) yield  $dp + q + h = dq + p + h$ , a contradiction and (10) follows.

The proof will be done in three cases; Case I.  $p < q$ , Case II.  $p = q$  and  $a \neq 1$ , Case III.  $p = q$  and  $a = 1$ .

**Case I.** Let  $p < q$ . We have  $q \geq 2$  by (10) and  $2q = \deg A + r + dq$  by (8). This with (10) implies  $\deg A = 0 = r$ , and  $d = 2$ . Hence  $A(z) = 1$  and  $R(z) = 1$ , since  $A(0) = 1$  by (7) and  $R(0) = 1$  by (8). Then we have by (8)

$$Q(z)^2 = Q(z^2) - azQ(z). \quad (11)$$

Writing  $Q(z) = a_q z^q + a_{q-s} z^{q-s} + \dots$ , where  $a_q \neq 0$ ,  $a_{q-s} \neq 0$  ( $1 \leq s \leq q$ ), we have from (11)

$$\begin{aligned} a_q^2 z^{2q} + 2a_q a_{q-s} z^{2q-s} + \dots \\ = a_q z^{2q} + a_{q-s} z^{2q-2s} + \dots - az(a_q z^q + a_{q-s} z^{q-s} + \dots). \end{aligned}$$

We see that  $a_q = 1$ . First we consider the case of  $q \geq 3$ . If  $1 \leq s \leq q - 2$ , then  $2q - s > 2q - 2s$  and  $2q - s > q + 1$ , so we have  $a_{q-s} = 0$ , which is a contradiction. Therefore we have  $s = q - 1$  or  $s = q$ . Thus we have  $Q(z) = z^q + a_1 z + 1$ , where  $a_1 \neq 0$  if  $s = q - 1$ ,  $= 0$  if  $s = q$ . We have from (11)

$$\begin{aligned} z^{2q} + 2a_1 z^{q+1} + 2z^q + a_1^2 z^2 + 2a_1 z + 1 \\ = z^{2q} + a_1 z^2 + 1 - az(z^q + a_1 z + 1). \end{aligned}$$

Noting that  $q \geq 3$  and comparing the coefficients of  $z^q$  in the both sides, we have a contradiction.

Therefore we have  $q = 2$ , and so  $Q(z) = z^2 + a_1 z + 1$ . It follows from (11) that

$$\begin{aligned} z^4 + 2a_1 z^3 + 2z^2 + a_1^2 z^2 + 2a_1 z + 1 \\ = z^4 + a_1 z^2 + 1 - az(z^2 + a_1 z + 1). \end{aligned}$$

Comparing the coefficients of the both sides, we have

$$2a_1 = -a, \quad 2 + a_1^2 = a_1 - aa_1.$$

Hence we have  $(a, a_1) = (4, -2)$  or  $(-2, 1)$ , and so we get

$$f(z) = \sum_{k \geq 0} \frac{4^k z^{2^k}}{(1 + z^{2^k})^2} = \frac{z}{(1 - z)^2}$$

$$f(z) = \sum_{k \geq 0} \frac{(-2)^k z^{2^k}}{1 - z^{2^k} + z^{2^{k+1}}} = \frac{z}{1 + z + z^2},$$

which are the rational functions given in the case (i) and (ii), respectively.

**Case II.** Let  $p = q$  and  $a \neq 1$ . We have from (8)  $2q = \deg A + r + dq$ . This with (10) implies  $\deg A = 0 = r, q = 1$ , and  $d = 2$ . Hence  $A(z) = 1$  and  $R(z) = 1$ , since  $A(0) = 1$  by (7) and  $R(0) = 1$  by (8). Writing  $Q(z) = 1 - bz$  with  $b \neq 0$ , we have from (8)

$$1 - 2bz + b^2 z^2 = 1 - bz^2 - az(1 - bz).$$

Comparing the coefficients of both sides, we have

$$b^2 = -b + ab, \quad 2b = a.$$

Hence we have  $a = 2, b = 1$ , and so we get

$$f(z) = \sum_{k \geq 0} \frac{2^k z^{2^k}}{1 + z^{2^k}} = \frac{z}{1 - z}.$$

which is the rational function given in the case (iii).

**Case III.** Let  $p = q$  and  $a = 1$ . From (8) we have

$$Q(z)^2 = A(z)\{R(z)Q(z^d) - z^{d-1}R(z^d)Q(z)\}. \quad (12)$$

**Lemma 2.1.** *We can express  $Q(z)$  as*

$$Q(z) = \prod_{i=1}^{d-1} (1 - \gamma_i^{-1} z)^{n_i} Q_1(z),$$

where  $\gamma_i$  ( $1 \leq i \leq d-1$ ) are the  $(d-1)$ -th roots of unity,  $n_i \geq 1$  ( $1 \leq i \leq d-1$ ), and  $Q_1(z) \in C[z]$  such that  $Q_1(\gamma_i) \neq 0$  for any  $i$ . Furthermore

$$A(z) = \prod_{i=1}^{d-1} (1 - \gamma_i^{-1} z)^{n_i} A_1(z),$$

where  $A_1(z) \in C[z]$  such that  $A_1(\gamma_i) \neq 0$  for any  $i$ . In particular,

$$d-1 \leq \deg A \quad \text{and} \quad d-1 \leq q. \quad (13)$$

**Proof.** Letting  $z = \gamma_i$  in (12) we have  $Q(\gamma_i) = 0$  for any  $i$ . We may put

$$Q(z) = \prod_{i=1}^{d-1} (1 - \gamma_i^{-1} z)^{n_i} Q_1(z),$$

where  $n_i \geq 1$  ( $1 \leq i \leq d-1$ ) and  $Q_1(z) \in C[z]$  such that  $Q_1(\gamma_i) \neq 0$  for any  $i$ . From (12) we have

$$\prod_{i=1}^{d-1} (1 - \gamma_i^{-1} z)^{n_i} Q_1(z)^2 = A(z) \{R(z) Q_1(z^d) \prod_{i=1}^{d-1} \varphi(\gamma_i^{-1} z)^{n_i} - z^{d-1} R(z^d) Q_1(z)\},$$

where  $\varphi(z) = (1 - z^d)/(1 - z)$ . Letting  $z = \gamma_j$  for fixed  $j$ , we have

$$0 = A(\gamma_j) R(\gamma_j) Q_1(\gamma_j) (\prod_{i=1}^{d-1} \varphi(\gamma_i^{-1} \gamma_j)^{n_i} - 1).$$

We note that  $\varphi(\gamma_i^{-1} \gamma_j) = 1$  if  $i \neq j$  and  $\varphi(\gamma_i^{-1} \gamma_j) = d$  if  $i = j$ . So  $\prod_{i=1}^{d-1} \varphi(\gamma_i^{-1} \gamma_j)^{n_i} - 1 = d^{n_j} - 1 \neq 0$ . Since  $R(\gamma_j) Q_1(\gamma_j) \neq 0$ , we obtain  $A(\gamma_j) = 0$  for any  $j$ . Therefore we may put

$$A(z) = \prod_{i=1}^{d-1} (1 - \gamma_i^{-1} z)^{n_i} A_1(z),$$

where  $A_1(z) \in C[z]$  such that  $A_1(\gamma_i) \neq 0$  for any  $i$ . The proof of the lemma is completed.

Now we return to the proof in Case III. It follows from (9) and (13) that

$$1 \leq \max\{d-1, \frac{h}{d+1}\} \leq q \leq \frac{h}{d-1} - 1. \quad (14)$$

In particular, we have

$$2 \leq d(d-1) \leq h. \quad (15)$$

In the case of  $h = 2$ , we have  $d = 2$  by (15) and so  $q = 1$  by (14). We have  $R(z) = 1$  by (10) and  $Q(z) = 1 - z$  by Lemma 2.1, which implies  $A(z) = 1 - z$  by (12), and so  $H(z) = 1 - z^2$  by (7). This gives the functional equation

$$f(z) = \sum_{k \geq 0} \frac{z^{2^k}}{1 - z^{2^{k+1}}} = \frac{z}{1 - z},$$

which is the rational function given in the case (iv).

If  $h = 3$ , we have  $d = 2$  by (15), and hence  $q = 1$  or  $2$  by (14). Assume that  $q = 1$ . Then we have  $Q(z) = 1 - z$ ,  $R(z) = 1$ , and  $A(z) = 1 - z$  by (12), which contradicts (9). If  $q = 2$ , we have  $Q(z) = (1 - z)(1 - bz)$ , where  $b \neq 1$ ,  $A(z) = 1 - z$ , and  $R(z) = 1 - cz$ , which implies  $(1 - bz)^2 = (1 - cz)(1 + z)(1 - bz^2) - z(1 - cz^2)(1 - bz)$ . Letting  $z = 1$  we have  $b = c$  since  $b \neq 1$ , which is impossible since  $P, R$  are coprime. The proof of Theorem 1.1 is completed.

## 2.2 Proof of Theorem 1.2

The proof is the same as these of Case I and II in Theorem 1.1, since the condition  $\deg H \leq 3$  is not used there.



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